The Variations of Parameters Kernel

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Define the function

$$g(x) = \begin{cases} 0 & x \notin [0,1] \\ 1 & x \in [0,1] \end{cases}$$

We are interested in the function F defined by

$$F(t) = \int_0^t \sin(t-s)g(s)ds.$$

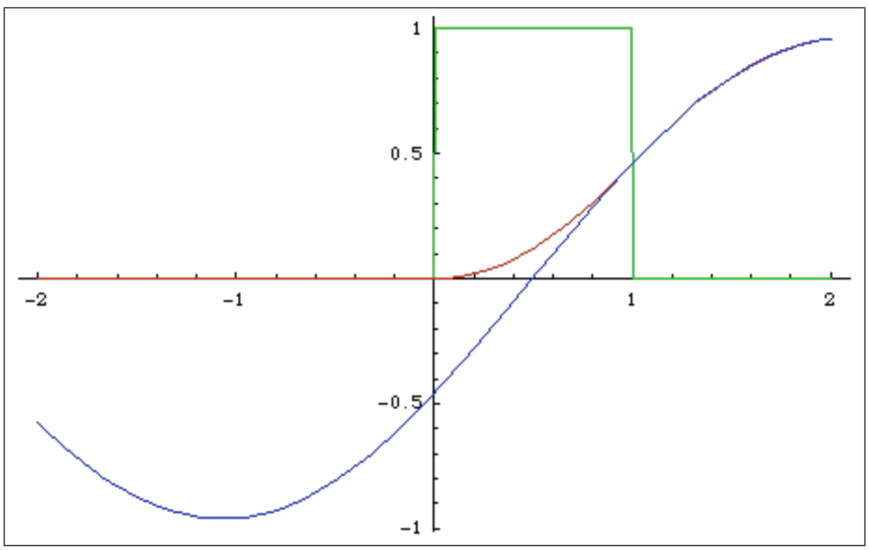
Compute as follows. If $t < 0$, then $F(t) = 0$ since $g = 0$ on the region of integration $[t, 0]$. If $t \in [0,1]$, we have

$$F(t) = \int_0^t \sin(t-s)ds = \cos(t-s)|_{s=0}^t = 1 - \cos(t)$$

since $g = 1$ in the region $[0,t]$. Now, for $t > 1$, we have

$$F(t) = \int_0^1 \sin(t-s)ds = \cos(t-s)|_{s=0}^1 = \cos(t-1) - \cos(t),$$

again because $g = 0$ outside $[0,1]$.
Here I've plotted g in green, F in red, and $\cos(t-1) - \cos(t)$ in blue.



The function F comes up in the following problem:

$$\boxed{y'' + y = g, \quad y(0) = 0, \; y'(0) = 0.}$$

Taking solutions $y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$, we get the Wronskian determinant $W(y_1, y_2; t) = 1$ and the variation of parameters formula gives

$$y(t) = \int^t [y_1(s)y_2(t) - y_1(t)y_2(s)]g(s)ds = \int^t [\cos(s)\sin(t) - \cos(t)\sin(s)]g(s)ds$$

$$= \int^t \sin(t-s)g(s)ds.$$

Describe the physical situation typically modelled by such an IVP.