INTEGRALS FROM ELLIPSES

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1. Arclength of an Ellipse

Parametric expression of ellipse $b > 0$, $\gamma(t) = \langle \cos(t), \sqrt{b}\sin(t) \rangle$.

Length:

$$l(b) = \int_0^{2\pi} |\gamma'(t)|dt = \int_0^{2\pi} \sqrt{\sin^2(t) + b\cos^2(t)}\ dt$$

Do a Taylor development of $l(b)$ at $b = 1$.

$$l(b) = \sum_{k=0}^{\infty} \frac{\frac{d^k}{db^k} \int_0^{2\pi} \sqrt{\sin^2(t) + b\cos^2(t)}|_{b=1}\ dt}{k!}(b-1)^k$$

Since the integrand is well-behaved, we may differentiate under the integral sign, obtaining

$$l(b) = \sum_{k=0}^{\infty} \frac{\int_0^{2\pi} \frac{\partial^k}{\partial b^k} \sqrt{\sin^2(t) + b\cos^2(t)}|_{b=1}\ dt}{k!}(b-1)^k$$

Compute

$$\frac{\partial^k}{\partial b^k}\sqrt{\sin^2(t) + b\cos^2(t)} = \begin{cases} (-1)^{k+1}\frac{(2k-3)!!}{2^k}\frac{\cos^{2k}(t)}{(\sin^2(t)+b\cos^2(t))^{(2k-1)/2}} & k > 1 \\ \frac{1}{2}\frac{\cos^{2k}(t)}{(\sin^2(t)+b\cos^2(t))^{(2k-1)/2}} & k = 1 \end{cases}$$

where for n odd, $n!! \stackrel{\text{def}}{=} n(n-2)(n-4)\dots 3 \cdot 1$.

Next, for $k > 1$ we get

$$a_k \stackrel{\text{def}}{=} \int_0^{2\pi} \frac{\partial^k}{\partial b^k}\sqrt{\sin^2(t) + b\cos^2(t)}|_{b=1}\ dt = (-1)^{k+1}\frac{(2k-3)!!}{2^k}\int_0^{2\pi} \frac{\cos^{2k}(t)}{(\sin^2(t)+\cos^2(t))^{(2k-1)/2}}\ dt$$

$$= (-1)^{k+1}\frac{(2k-3)!!}{2^k}\int_0^{2\pi}\cos^{2k}(t)\ dt = (-1)^{k+1}\frac{(2k-3)!!}{2^k}\int_0^{2\pi}\cos^{2k}(t)\ dt$$

Raising the identity $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$ to powers and iterating, we see that this last integral has values

TABLE 1

k	$\int_0^{2\pi} \cos^{2k}(t)\ dt$
0	2π
1	π
2	$3\pi/4$
3	$5\pi/8$
4	$35\pi/64$
5	$63\pi/128$
6	$231\pi/512$
7	$429\pi/1024$
8	$6435\pi/16384$
9	$12155\pi/32768$

Thus

$$l(b) = 2\pi + \frac{\pi}{2}(b-1) + \sum_{k=2}^{\infty} \left[(-1)^{k+1} \frac{(2k-3)!!}{2^k k!} \int_0^{2\pi} \cos^{2k}(t) dt \right] (b-1)^k$$

Mathematica informs us that if E is the elliptic integral defined so,

$$E(\phi, m) = \int_0^{\phi} \sqrt{1 - m\sin^2(\theta)} d\theta, \quad E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2(\theta)} d\theta$$

then

$$l(b) = \int_0^{2\pi} \sqrt{\sin^2(t) + b\cos^2(t)}\ dt = 2\left(E[1-b] + \sqrt{b} E\left[\frac{b-1}{b}\right]\right).$$

Stupid observation: $\lim_{b\to\infty} l(b)/\sqrt{b} = 4$.

Does $l(b)$ satisfy an interesting ode?

2. Ellipsoids

Parametrization: An ellipsoid is given by $\gamma(s,t) = \langle \sin(s)\cos(t), \sqrt{b}\sin(s)\sin(t), \sqrt{c}\cos(s)\rangle$, $s \in [0,\pi]$, $t \in [0, 2\pi]$, $b, c > 0$. Its surface area is

$$A(b,c) = \int_0^{2\pi} dt \int_{-1}^{1} ds |\gamma_s \times \gamma_t|$$

We compute

$$\gamma_s \times \gamma_t = \langle \sqrt{bc}\sin^2(s)\cos(t), \sqrt{c}\sin^2(s)\sin(t), \sqrt{b}\sin(s)\cos(s)\rangle$$

$$A(b,c) = \int_0^{2\pi} dt \int_{-1}^{1} ds \sqrt{bc\sin^4(s)\cos^2(t) + b\sin^2(s)\cos^2(s) + c\sin^4(s)\sin^2(t)}$$

Taylor series:

$$A(b,c) = \sum_{j,k=0}^{\infty} \frac{1}{j!k!} \frac{\partial^{j+k} A}{\partial b^j \partial c^k}\bigg|_{(1,1)} (b-1)^j (c-1)^k$$