THE INVARIANCE OF THE PATH INTEGRAL UNDER REPARAMETRIZATION

Definition: If f is a complex-valued function and $\gamma \subset \mathbb{C}$ is a path, then we define

$$\int_\gamma f \;=\; \int_a^b f(z(t))z'(t)dt$$

for *any* parametrization $z : [a,b] \to \mathbb{C}$ of γ.

Remark: Notice that the symbol $\int_\gamma f$ makes no reference to the particular parametrization chosen for γ. The reason is that the definition has been formulated exactly in such a way that it be irrelevant. This irrelevance is referred to as the invariance of the integral under reparametrization. As it is written, it also could be referred to by saying that $\int_\gamma f$ is *well-defined*, meaning that all the things on which it depends are included in the symbol–notably the choice of parametrization has been ignored.

As this is a recurring theme in math, I will include another example here. In linear algebra, an important functional from the set of linear transformations L to the coefficient field (usually $\mathbb{R}$ or $\mathbb{C}$) is the trace, denoted tr. The usual way to compute the trace of a transformation given abstractly is to choose an arbitrary basis e_k of the domain of L (analogously to our choosing a parametrization of γ arbitrarily) and perform the sum $\sum_k < Le_k, e_k >$, ($<,>$ is the inner product in the range of L). The symbol for trace, tr, does not make reference to the basis e_k with respect to which it is taken because it does not depend on it in fact. Why this is so can be seen as follows: If d_k is any other basis for the domain of L, Dom$L$, then it is related to e_k by an invertible matrix M acting from Dom$L$ to Dom$L$ by $d_k = \sum_l M_{kl} e_l$, thus the trace taken with respect to d_k is $\sum_k < Ld_k, d_k >= \sum_{klm} < LM_{kl}e_l, M_{km}e_m >= \sum_{klm} < M_{kl}Le_l, M_{km}e_l >= \sum_{klm} < M^\dagger_{km} M_{kl} Le_l, e_m >= \sum_{lm} < \delta_{ml} Le_l, e_m >= \sum_l < Le_l, e_l >$ and we have invariance here too.

Coming back to the integrals, the following would establish the invariance:

Claim: If $\zeta : [a,b] \to \gamma$ and $\xi : [c,d] \to \gamma$ are arbitrary parametrizations of γ then

$$\int_a^b f(\zeta(t))\zeta'(t)dt \;=\; \int_c^d f(\xi(\tau))\xi'(\tau)d\tau$$

Proof: For simplicity, let us first assume that ζ is a one-to-one function onto γ. Then there exists a function $t(\tau)$ so that $\zeta(t(\tau)) = \xi(\tau)$ since ζ is invertible. For this situation, we have the range of the function t equal $[a,b]$ and $\zeta'(t(\tau))t'(\tau) = \xi'(\tau)$ by the chain rule. Further we have

$$f(\zeta(t(\tau))) \;=\; f(\xi(\tau)) \quad \text{so} \quad f(\zeta(t(\tau)))\zeta'(t(\tau))t'(\tau) \;=\; f(\xi(\tau))\xi'(\tau).$$

Integrating both sides with respect to τ between c and d,

$$\int_c^d f(\zeta(t(\tau)))\zeta'(t(\tau))t'(\tau)d\tau \;=\; \int_c^d f(\xi(\tau))\xi'(\tau)d\tau.$$

But the usual formula for substitutions in Calculus II,

$$\int_{u(a)}^{u(b)} f(u)du \;=\; \int_a^b f(u(t))u'(t)dt,$$

tells us the left-hand side is just $\int_a^b f(\zeta(t))\zeta'(t)dt$.

Now, again using integration by substitution, we arrive, as does Ahlfors, at the fact $\int_{-\gamma} f \;=\; -\int_\gamma f$ (this is clear because as in the definition, $-\gamma$ would be traversed by $z(t)$ with t varying backwards from b to a). This being the case, any continuously differentiable parametrization of γ can be broken up into pieces in which its direction is constant. On those pieces where the parametrization is not one-to-one, it is easy to see that all the motion cancels out except the one in the direction of the overall orientation of γ. Thus we have the claim established for all continuously differentiable parametrizations, ono-to-one or not.

Examples: For $f(z) = z^2$, γ the line segment connecting 0 with $1+i$, choose parametrization $z(t) = (1+i)t,\ 0 \le t \le 1$, then $\int_\gamma f \;=\; \int_0^1 [(1+i)t]^2[(1+i)t]'dt \;=\; (1+i)^3 \int_0^1 t^2 dt \;=\; \frac{(1+i)^3}{3}$.

For $f(z) = z^2$, γ the unit circle centered at the origin, choose parametrization $z(t) = e^{it},\ 0 \le t \le 2\pi$, then $\int_\gamma f \;=\; \int_0^{2\pi} [e^{it}]^2[e^{it}]'dt \;=\; i\int_0^{2\pi} e^{3it}dt \;=\; 0$.

For $f(z) = \frac{1}{z}$, γ the unit circle centered at the origin, and the same parametrization as before, $\int_\gamma f \;=\; \int_0^{2\pi} [e^{-it}][e^{it}]'dt \;=\; i\int_0^{2\pi} dt \;=\; 2\pi i$.

For $f(z) = \frac{1}{z}$, γ the circle of radius r centered at the origin. Then $\int_\gamma f \;=\; 2\pi i$. *You show this.*

For $f(z) \;=\; \frac{1}{z}$, γ the square of side-length 1, centered at the origin, you complete the following: I will do the right wall. $z(t) = \frac{1}{2} + it$ so the contribution is $\int_{-1/2}^{1/2} \frac{1}{\frac{1}{2}+it}[\frac{1}{2}+it]'dt \;=\; i\int_{-1/2}^{1/2} \frac{dt}{\frac{1}{2}+it} \;=\; 2i\int_{-1/2}^{1/2} \frac{1-2it}{1+4t^2}dt$ which you should have no trouble doing. The answer for the whole thing is $2\pi i$ just like for the circle giving a very interesting property of the path integral.

*Old Junk**: A solution to the exercise in the previous handout is the following. The series

$$\sum_{n=1}^{\infty} 2^{(-1)^n - n} \;=\; \frac{1}{2^2} + \frac{1}{2^1} + \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^5} + \cdots$$

is convergent as it is a rearrangement of the absolutely convergent $\sum_n 2^{-n}$. A refined form of the ratio test is

$$\overline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \;<\; 1 \;\Rightarrow\; \sum a_n < \infty$$

$$\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \;>\; 1 \;\Rightarrow\; \sum a_n = \infty.$$

For our example, we have $\overline{\lim}\left|\frac{a_{n+1}}{a_n}\right| = 2$, from which we can deduce nothing, and $\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| = 1/8$, also inconclusively. However,

$$\sqrt[n]{a_n} = 2^{\frac{(-1)^n - n}{n}} \to 2^{-1} \text{ as } n \to \infty$$

predicting, correctly the convergence of $\sum_{n=1}^{\infty} 2^{(-1)^n - n}$.

Another bit of old junk: The translation of results about series to ones about power series. We have

$$\overline{\lim}\sqrt[n]{|b_n|} < 1 \Rightarrow \sum |b_n| < \infty$$

replacing $b_n \to a_n z^n$, we obtain

$$\overline{\lim}\sqrt[n]{|a_n z^n|} = |z|\ \overline{\lim}\sqrt[n]{|a_n|} = |z|\frac{1}{R} < 1 \Rightarrow \sum |a_n z^n| < \infty$$

but $|z|\frac{1}{R} < 1 \Leftrightarrow |z| < R$.

If it works substantially differently for the ratio test, $\Rightarrow$ I am a monkey's uncle.

* Reference: *Counterexamples in Analysis* Gelbaum & Olmsted.