MATH 362 EXAM I

1) Evaluate $\sqrt{i}$ and find the four roots of the equation $z^4 + 1 = 0$.

2) Show that the map $\phi : \mathbb{C} \to \mathrm{Mat}_{\mathbb{R}}(2,2)$ (the 2×2 real matrices) given by

$$\phi : \alpha + i\beta \mapsto \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

is an algebra homomorphism. That is, for complex numbers z_1, z_2 show that ϕ is additive and multiplicative: $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ and $\phi(z_1 z_2) = \phi(z_1)\phi(z_2)$. Here, the plus between complex numbers means addition of complex numbers while that between matrices means addition of matrices just as juxtaposition of complex numbers means multiplication and that of matrices means multiplication of matrices. Which complex numbers does ϕ send to singular matrices? Consider that for a complex-valued function to be differentiable and satisfy the usual real function relation $f(z) \approx f(a) + f'(a)(z - a)$, the derivative of f as a function from $\mathbb{R}^2$ to itself has a representation as a 2×2 matrix. For this to jibe with the above equation, there must be relations among the elements of the matrix $f' = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. These are precisely the Cauchy-Riemann equations. This gives another way of proving them. By considering what the mapping ϕ does to $f(z) \approx f(a) + f'(a)(z - a)$, prove the Cauchy-Riemann equations.

3) Verify the Cauchy-Riemann equations being solved by $f(z)$ exactly when they are by $\overline{f(\overline{z})}$. *Hint:* Write $f(z) = u(x, y) + iv(x, y)$. Now notice that $f(\overline{z}) = u(x, -y) + iv(x, -y)$ (*why?*). Now repeat the process for the big conjugation and write down the Cauchy-Riemann equations in both cases. They should be the same. Suppose $f(z) = \sum_{k=0}^{\infty} a_k z^k$. What is the expansion for $\overline{f(\overline{z})}$? This would be another way of proving the statement if we knew each analytic f had an expansion in power series. Do we know that already?

4) Find the radius of convergence for the following power series:
i) $\sum n^p z^n$,
ii) $\sum \frac{z^n}{n!}$,
iii) $\sum n! z^n$.

5) Evaluate

$$\lim_{p \to \infty} \sqrt[p]{a^p + b^p}.$$

6) Compute $\cos(i)$.

7) Determine all the values of $2^i,\ i^i,\ (-1)^{2i}$.

8) Compute $\int_{|z|=2} \frac{dz}{z^2 - 1}$. The circle is traversed in a counterclockwise sense. Use Cauchy–you'll be happy you did.

9) Show that $\int_\gamma P(z)d\bar{z} = -2\pi i R^2 P'(a)$ where γ is the circle $|z-a| = R$ taken counterclockwise.

10) Compute
$\int_{|z|=1} e^z z^{-n} dz$,
$\int_{|z|=2} z^n (1-z)^m dz$

11) Prove that any analytic function satisfying $|f(z)| \leq |z|^n$ for some n and all z sufficiently large must reduce to a polynomial.