IN-CLASS EXERCISE IV

Today we will go through some beautiful stuff on pp 63-65 of Rudin's book. It makes rigorous some stuff you already know about e and you will learn why e is irrational, too.

First, recall from the Taylor series section in Calculus that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \tag{1}$$

This gives us a natural *definition* for e (notice we don't know Taylor's theorem yet!).

$$e = e^1 = \sum_{k=0}^{\infty} \frac{1}{k!} \tag{2}$$

Now that we are talking about series with positive terms we need only argue that the partial sums be bounded in order to conclude convergence (Thm 3.24 in Rudin). Thus:

$$e = \frac{1}{1} + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots + \frac{1}{n!} + \cdots \tag{3}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} \tag{4}$$

Why?

This second sum is bounded by 3.

Why? (See p 61 of Rudin's book for a hint.)

Theorem:(3.31 Rudin) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$

Proof: First let's recall the binomial theorem (Exercise: you prove it by induction: For $p = 1$ it is trivial, you proceed.):

$$(a+b)^p = \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} a^{p-k} b^k \tag{5}$$

Applying this to the exression $\left(1 + \frac{1}{n}\right)^n$ we obtain

$$t_n \stackrel{\text{def}}{=} \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(\frac{1}{n}\right)^k = \tag{6}$$

(7)
$$= \frac{n!}{0!(n-0)!}\left(\frac{1}{n}\right)^0 + \frac{n!}{1!(n-1)!}\left(\frac{1}{n}\right)^1 + \frac{n!}{2!(n-2)!}\left(\frac{1}{n}\right)^2 + \frac{n!}{3!(n-3)!}\left(\frac{1}{n}\right)^3 + \cdots + \frac{n!}{n!(n-n)!}\left(\frac{1}{n}\right)^n$$

(8)
$$= 1 + \frac{n}{n} + \frac{n(n-1)}{2!n^2} + \frac{n(n-1)(n-2)}{3!n^3} + \frac{n(n-1)(n-2)(n-3)}{4!n^4} + \cdots + \frac{n(n-1)(n-2)(n-3)\cdots(1)}{n!n^n}$$

(9)
$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right) + \cdots +$$

(10)
$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

So we conclude that

(11)
$$t_n = \left(1+\frac{1}{n}\right)^n < \sum_{k=0}^{n} \frac{1}{k!}$$

Why?

and so the biggest limit point $\bar{t}$ of the set $\{\left(1+\frac{1}{n}\right)^n | n \in \mathbb{N}\}$ satisfies $\bar{t} \le e$. **Why?**

Now notice that if $n \ge m$

(12)
$$t_n = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots +$$

(13)
$$\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

(14)
$$\ge 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \stackrel{\text{def}}{=} A_{n,m}$$

Why?

Now notice that the smallest limit point $\underline{t}$ of the t_n must satisfy

(15)
$$\underline{t} \ge \lim_{n\to\infty} A_{n,m} = \sum_{k=0}^{m} \frac{1}{k!}.$$

Double why?

Letting $m \to \infty$ we get that $\bar{t} \le e \le \underline{t}$ and so the theorem is proven. •

Now let's prove something you don't know: The first point is that the partial sums

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \tag{16}$$

converge to e very rapidly. You've heard that, but let's be precise (following Rudin p 65).

$$e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) \tag{17}$$

Why?

and so

$$e - s_n =< \frac{1}{n\ n!} \tag{18}$$

Why?

so $0 < e - s_n =< \frac{1}{n\ n!}\ \forall n$.

Theorem: e is irrational**!**

Proof: Suppose not. Then there are $p, q \in \mathbb{N}$ so that $e = p/q$. Now

$$0 < q!(e - s_q) < \frac{1}{q}. \tag{19}$$

Why?

By assumption, we have that $qe = p \in \mathbb{N}$ and so $q!e \in \mathbb{N}$,too. Observe that

$$q!s_q = q! \sum_{k=0}^{q} \frac{1}{k!} \tag{20}$$

is an integer. **Why?**

Then $q!(e - s_q)$ must be an integer in $(0, 1)$, $\rightarrow\leftarrow$. $\bullet$

It is also known that e is not algebraic. This I will demonstrate in Analysis II as it is helpful to know a little about the Γ function. The proof is in M Spivak's excellent book *Calculus*.

Math trivia: It is unknown whether $\pi + e$ is rational!