IN-CLASS EXERCISE II

In the previous in-class exercise we demonstrated that if $f : \mathbb{R} \to [0, \infty)$ is a function satisfying the following three properties

- $f(0) = 0$
- $x < y \Rightarrow f(x) < f(y)$ (f is strictly increasing)
- $f(x + y) \leq f(x) + f(y)$ (f is subadditive),

we could conclude that $D(x, y) = f(d(x, y))$ is a metric on X whenever d is.

1) Now let's apply our theorem: Assuming we know that on $\mathbb{R}$, the function $d(x, y) = |x - y|$ defines a metric (weve seen this in class), you deduce that

- $D_1(x, y) = \frac{|x-y|}{1+|x-y|}$
- $D_2(x, y) = \ln(1 + |x - y|)$

are metrics on $\mathbb{R}$. Notice that for D_1 no two points of $\mathbb{R}$ are more than distance 1 from each other. You should contemplate the definition of boundedness now.

2) Next, we discuss the significance of compactness. Recall that in class I mentioned that compact sets are to continuous functions as finite sets are to arbitrary functions. More specifically, if f is any real-valued function on a finite set, f is bounded. If f is a continuous function defined on a compact set, then the same is true. Let's prove that.

First we'll need to cheat a little and remember from Calculus I that a function f is continuous at a point x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0), \tag{1}$$

or in words, limits and evaluation commute at x_0. Another thing we need is the definition of a function's continuity on a whole set. That just means that the x_0 above could have been anything in the set were talking about f's being continuous on. We should also remark that the assumptions of continuity and compactness are necessary.

You come up with counterexamples to the following (false) statements:

- Any function $f : E \to \mathbb{R}$ is bounded if $E \subset \mathbb{R}$ is compact.

- If $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$ is continuous, it is bounded.

Let us recall the relevant theorem from the text:

Theorem[Rudin 2.41] For $E \subset \mathbb{R}^k$, the following are equivalent:

- E is closed and bounded
- E is compact
- Every infinite subset of E has a limit point in E

We want:

Theorem[Us, now] If $f : E \to \mathbb{R}$, then f is bounded above whenever f is continuous and E is compact.

Proof: First note that it is enough to get just the max because the min is just the negative of the max of $-f$.

Our usual method of proof when we can't construct much or have to deal with all kinds of *whatever* is to suppose not-that way we have one thing in our hands.

Suppose then that f is unbounded on E.

- This gives rise to an infinite subset of E (How?)

- which then has a limit point in E. (Why?)

- So what?