IN-CLASS EXERCISE III

First a homework assignment: On p 78 of Rudin do 1, 2, 3, 6, 9.

Here in class we'll do problem 7 after completing last time's example:
Claim: $a_k = 1/k$ is convergent.
Proof: For $\epsilon > 0$ we want n, m large enough so that $|\frac{1}{m} - \frac{1}{n}| = |\frac{n-m}{mn}| < \epsilon$. Simply notice that

$$|\frac{n-m}{mn}| < \frac{n}{mn} = \frac{1}{m} \tag{1}$$

and this can be made as small as we please by Rudin's theorem 1.20.

Now for Rudin's stuff:

7)[Rudin p 78] $a_n \geq 0 \ \forall n \ \Rightarrow (\sum a_n < \infty \Rightarrow \sum \frac{\sqrt{a_n}}{n} < \infty)$.

With theorem 3.11 of Rudin's book we translate the problem to

$$(\forall \epsilon > 0 \ \exists N = N_\epsilon, n \geq N \Rightarrow a_n + a_{n+1} + \cdots + a_{n+k} < \epsilon, \ \forall k) \tag{2}$$

$$\Rightarrow (\forall \delta > 0 \ \exists M = M_\delta, n \geq M \Rightarrow \frac{\sqrt{a_n}}{n} + \frac{\sqrt{a_{n+1}}}{n+1} + \cdots + \frac{\sqrt{a_{n+k}}}{n+k} < \delta, \ \forall k) \tag{3}$$

Now, **POOF!** if we put

$$\alpha_k = \frac{1}{k} \quad \text{and} \quad \beta_k = \sqrt{a_k} \tag{4}$$

we have

$$\sum_{k=1}^{n} \alpha_k \beta_k \leq \sqrt{\sum_{k=1}^{n} \alpha_k^2} \sqrt{\sum_{k=1}^{n} \beta_k^2} \tag{5}$$

Why?

Complete the proof.