ANALYSIS I FINAL EXAM

Work singly and stay away from the Net and books other than Folland's. For all problems, either prove the claims or provide a counterexample.

1. The Structure of $\mathbb{R}^n$

Claim 1.1. *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ *if and only if* $\mathbf{x} \cdot \mathbf{y} = 0$.

Claim 1.2. *Let* $\mathbb{Q}$ *denote the rational numbers and consider* $\mathbb{Q} \subset \mathbb{R}$. *Then* $\mathbb{Q}^{\text{int}} = \emptyset$ *but* $(\overline{\mathbb{Q}})^{\text{int}} = \mathbb{R}$.

Claim 1.3. *There is no subset* $A \subset \mathbb{R}$ *that is both open and closed. Furthermore, there is no subset* $A \subset \mathbb{R}$ *satisfying* $(\overline{A})^{\text{int}} = \overline{A^{\text{int}}}$.

2. Limits and Continuity

Claim 2.1. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *the function* $f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ *is continuous.*

Claim 2.2. *For positive* $x \in \mathbb{R}$, *the function* $f(x) = \sin(1/x)$ *is continuous. Hint: You should use the definition of the* sin.

Claim 2.3. *Define the sequence* $a_k = \sin(k)$ *for* $k \in \mathbb{N}$. *Then* $\liminf a_k = -1$ *and* $\limsup a_k = 1$.

3. Compactness and Connectedness

For two sets $A, B \subset \mathbb{R}^n$ define their product by

$$A \times B \stackrel{\text{def}}{=} \{(x, y) | x \in A, y \in B\}.$$

Claim 3.1. *If* $A, B \subset \mathbb{R}^n$ *are compact, then* $A \times B \subset \mathbb{R}^{2n}$ *is compact.*

Claim 3.2. *If* $A, B \subset \mathbb{R}^n$ *are arcwise connected, then* $A \times B \subset \mathbb{R}^{2n}$ *is arcwise connected.*