MATH 262 TAYLOR SERIES REVIEW

Theorem: If f has a series representation at a, it is of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

with $f^{(k)}(a)$ the kth derivative of f, evaluated at a.

$T_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is usually called the Nth Taylor polynomial for f and $R_N(x) = \sum_{k=N+1}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the Nth remainder term. The remainder term tells you how good the approximation to f T_N is. Also, If you need to show that the series really converges to f on some interval $x \in [a,b]$, the following provides the means (a criterion) for establishing this (just get $|R_N(x)| \to 0$ as $N \to \infty$ for $x \in [a,b]$).

Theorem: (Taylor's Inequality) If $|f^{(N+1)}| \leq M$ for $|x-a| \leq d$, then

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{(N+1)}$$

Examples:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

and so

$$e^{ix} = \cos(x) + i\sin(x)$$

by direct calculation.

Exercises: Find the 3rd order Taylor polynomials for the following functions near the indicated points.

- $f(x) = \ln(1+x) \qquad a = 0$
- $f(x) = e^{x^2} \qquad a = 0$

Taylor series can be used to evaluate limits, too. Find $\lim_{x\to 0} \frac{\sin(x)-x}{x^2}$ and $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$ by any means you wish, but try by expanding the sin in its Taylor series and working out the computation.

Maple can do these for you immediately.