MATH 261 IN-CLASS EXERCISE I

Today, we'll discuss Green's Theorem and use it to compute a few integrals.
Theorem: If C is a *simple* (meaning it does not cross itself) *closed* (meaning it starts and ends at the same point) curve in the plane and D is the the region it surrounds, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \tag{1}$$

At first glance we can already discern something. It is that the condition for path-independence/conservatism involves the integrand on the right. Recall that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

implies path-independence for the vectorfield $\mathbf{F}(x, y) = \langle P, Q \rangle$. But the path has to be closed for the theorem to be in force so Green says what we already know: conservative vectorfields give zero for loop integrals.

This is the second form of the fundamental theorem of calculus you'll see in Calc III (the first was the FTC for line integrals). Notice that the left-hand side is a path-integral (single) while the right-hand side is an area integral of a kind of derivative (introduce the notation $\nabla\times$) of a vector-valued function $\mathbf{F} = \langle P, Q \rangle$; then the theorem reads

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \nabla \times \mathbf{F} \; dA. \tag{2}$$

This is not quite standard notation, but in the plane it is unambiguous. See Stewart p 957 eq 12 for the correct version.

To see why Green's theorem is so, it is clear that it is enough (linearity of the integral) to show that

$$\int_C P dx = \iint_D \left(-\frac{\partial P}{\partial y} \right) dA \quad \text{and} \quad \int_C Q dy = \iint_D \left(\frac{\partial Q}{\partial x} \right) dA. \tag{3}$$

for the regions and curves under consideration. What is usually done now is to take D and break it up into lots of little rectangles and triangles. Once one does this, it is clear that when adding up all these little shapes' contributions to build up the area integral the path integral's contribution from paths that are internal to D all cancel. This gives the result if we can just show that Green is true for an arbitrary triangle (we can build rectangles from two triangles).

Rather than doing that (which is a little more complicated) we'll content ourselves with the rectangle computation and a few examples–otherwise we'll never get to the big theorems at the end.

$$\int_{Box} Pdx = \iint_{Interior of Box} \left(-\frac{\partial P}{\partial y}\right) dA \tag{4}$$

is then what we'll show.

Let *Interior of Box* $= \{(x,y)|x \in [a,b] \ \&\ y \in [c,d]\}$, then

$$-\iint_{IoB} \left(\frac{\partial P}{\partial y}\right) dA = -\int_a^b dx \int_c^d dy \frac{\partial P}{\partial y} = -\int_a^b dx[P(x,d) - P(x,c)] \tag{5}$$

$$= -\int_a^b dx[P(x,d) - P(x,c)] = \int_a^b dx P(x,c) - \int_a^b dx P(x,d) = \int_{Box} Pdx \tag{6}$$

once one makes sense of the orientation (the counterclockwise loop integral gives the signs above) and that on the vertical parts of the path, $dx = 0$. Notice also that the fundamental theorem of calculus was used in the demonstration. **Where?** This will recur in the remainder of the course. The other integral is similar.

An application [prob 11, p 951, Stewart]:

$$\int_C y^3 dx - x^3 dy \tag{7}$$

with C the circle of radius 2 centered at the origin.

Solution: Green's theorem reads

$$\int_C y^3 dx - x^3 dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{8}$$

where D is the disc of radius 2 centered at the origin and $(P,Q) = (y^3, -x^3)$. So

$$\int_C y^3 dx - x^3 dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left(\frac{\partial(-x^3)}{\partial x} - \frac{\partial y^3}{\partial y}\right) dA \tag{9}$$

$$= -3 \iint_D (x^2 + y^2) dA = -3 \int_0^{2\pi} d\vartheta \int_0^2 r^3 dr. \tag{10}$$

For fun we can see what Green saved us. Try to do the path integral without the heavy machinery: Parametrizing the curve $t \mapsto (2\cos t, 2\sin t)$,

$$\int_C y^3 dx - x^3 dy = \int_0^{2\pi} \langle y^3, -x^3 \rangle \cdot \langle -2\sin t, 2\cos t \rangle dt \tag{11}$$

$$= \int_0^{2\pi} \langle 8\sin^3 t, -8\cos^3 t \rangle \cdot \langle -2\sin t, 2\cos t \rangle dt = -16 \int_0^{2\pi} (\sin^4 t + \cos^4 t) dt. \tag{12}$$