MATH 261 FINAL EXAM

1) Set up any integral giving the volume of the solid above the cone $z = \sqrt{x^2+y^2}$, under the sphere $x^2+y^2+z^2=1$, in the first octant, and further than $1/2$ from the z-axis.

2) Reexpress the following integrals in cylindrical or spherical coordinates, whichever would give the simpler computation. Then do not compute them.

- $$\int_0^3 dy \int_0^{\sqrt{9-y^2}} dx \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} dz\ (x^2+y^2+z^2)$$
- $$\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{x^2+y^2}^{2-x^2-y^2} dz\ (x^2+y^2)^{3/2}$$

3) Use the transformation to evaluate the integral:

$$\iint_R x^2 dA \quad R = \{(x,y)|\ 9x^2+4y^2 \le 36\}; \quad x = 2u,\ y = 3v.$$

4) For $\mathbf{F}(x,y,z) =< x, -z, y >$, and C given by $\mathbf{r}(t) = (2t, 3t, -t^2),\ -1 \le t \le 1$, evaluate the line integral $\int_C \mathbf{F}{\cdot}d\mathbf{r}$. Is this vector field conservative?

5) Recall two forms of Green's theorem in the plane:

$$\oint_{\partial D} \mathbf{F}{\cdot}d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ dA \qquad \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\ ds = \iint_D \nabla \cdot \mathbf{F}\ dA.$$

Two related implications follow (if D is simply connected):

- $\nabla \times \mathbf{F} = 0 \Rightarrow \exists f\ s.t. \nabla f = \mathbf{F}$
- $\nabla \cdot \mathbf{F} = 0 \Rightarrow \exists \mathbf{G}\ s.t.\ \nabla \times \mathbf{G} = \mathbf{F}$

If possible, express $\mathbf{F}(x,y,z) =< yz, xz, xy >$ as a gradient or curl of something else (or both).

6) Compute the surface integral (recall that if the surface is given by $z = f(x,y)$, $dS = \sqrt{1+(\partial_x z)^2+(\partial_y z)^2} dA$)

$$\iint_S z\ dS, \quad S = \{(x,y,x^2+y^2)|\ x^2+y^2 \le 1\}.$$

7) Recall Stokes' theorem: $\oint_{\partial S} \mathbf{F}{\cdot}d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$. Compute $\oint_{\partial S} \mathbf{F}{\cdot}d\mathbf{r}$ for the $\mathbf{F}$ from problem 4 and the S from problem 6.

8) Recall the divergence theorem: $\iint_{\partial E} \mathbf{F}{\cdot}d\mathbf{S} = \iiint_E \nabla{\cdot}\mathbf{F}\ dV$. Compute $\iint_{\partial E} \mathbf{F}{\cdot}d\mathbf{S}$ for the $\mathbf{F}$ from problem 4 and E the region in the ellipsoid of problem 6.