MATH 251 FINAL EXAM

1) Compute

$$i)\ \begin{pmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 0 & 4 \end{pmatrix}.$$

$$ii)\ \begin{pmatrix} -2 & 4 \\ 0 & 4 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 5 & 6 \\ 2 & -4 & 0 \end{pmatrix}.$$

$$iii)\ \text{For } M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},\ MM^T.$$

2) For problem 1 *ii*, compute the determinant of the result. Why do you think it is what it is? For problem 1 *iii*, give a geometric interpretation (**a**, **b** vectors) of MM^T's being invertible.

3) For which values of κ, λ is the system

$$\kappa x + y = 0$$

$$x + \mu y = 1$$

solvable? *Do this by hand.*

4) Is it true that for any invertible M, $(M^{-1})^T = (M^T)^{-1}$? If so, prove it for 2×2 matrices. If not, give a counterexample.

5) Find all the 2×2 matrices commuting with

$$i)\ P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$ii)\ R_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad (\forall t \in \mathbb{R}).$$

Is there a condition on t so that $P_1 R_t = R_t P_1$? Explain the geometric meaning of these results.

6) Find a basis for the kernel of the following transformation.

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

Now find a basis for its range. Give its rank, dimension and nullity. I know this is too easy.

7) An *orthogonal* matrix M is one satisfying $M^T = M^{-1}$. Show that R_t, defined above, is orthogonal for all t. Show that all orthogonal matrices have determinant ± 1.

8) Show that the set of continuous real-valued functions on the real line form a subspace of the linear space of all real-valued functions on the real line.

9) Show that the derivative $\frac{d}{dx}$ is a linear transformation from the space of continuously differentiable functions to the continuous functions. Find a basis for the kernel of $\frac{d}{dx}$. *Hint: The kernel is one-dimensional.* Notice that, from the Fundamental Theorem of Calculus, $\frac{d}{dx}$ and $\int dx$ are nearly inverses: $\frac{d}{dx} \int dx = \text{id}$ but not the other way. They fail to legitimate inverses because of the nontrivial kernel you found in the first part. How does it show up in the integral? *Hint: Work out a simple problem of trying to invert a transformation with a small kernel.*

10) Find the eigenvalues, eigenvectors, and diagonalization of the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 5 & 2 \end{pmatrix}.$$

11) What is the geometric meaning of R_t's having the eigenvalues it does when $t \neq 0$? Show that $R_s R_t = R_t R_s = R_{t+s}$, $R_t^T = R_t^{-1} = R_{-t}$. What this means, TB?