The $\bar{\partial}$-Neumann Problem on Strongly Pseudoconvex G-Manifolds

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Abstract

Folland, Kohn, and Nirenberg's solution of the $\bar{\partial}$-Neumann problem, as presented in *The Neumann Problem for the Cauchy-Riemann Complex*, is valid for compact, strongly pseudoconvex complex manifolds. Here I announce a few results in a program to generalize as many of the results as possible from this work to the setting in which the strongly pseudoconvex complex manifold is a principal G-bundle with G unimodular Lie and the complex structure is invariant under the group's action.

Our techniques include hard analysis (to modify Folland and Kohn's *a priori* estimates), distributions, functional analysis (the spectral theorem), von Neumann algebras, and some techniques from harmonic analysis have turned out to be crucial.

This talk contains work by M. Engliš, M. Gromov, G. Henkin, M. Shubin, and the speaker.

1. History: Hartogs and Levi's problem

Geometrically describe the smooth regions $M \subset \mathbb{C}^n$ that are *domains of holomorphy*. *I.e.* regions with the property that for any point of the boundary, there is a holomorphic function which cannot be extended beyond the boundary at that point.

Condition:

If a region M is defined by a smooth real-valued function ρ by $M = \{z \in \mathbb{C}^n \mid \rho(z) < 0\}$, we say it is strongly pseudoconvex if at each point ξ of the boundary, the form

$$\left.\frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_l}\right|_{\xi}$$

is positive definite.

Alternatively we say that each point of the boundary of a strongly pseudoconvex region is a *peak point* for the holomorphic functions on M. Levi constructed holomorphic functions blowing up at any point of the such boundaries.

2. History: Levi's function

Suppose $bM = \{\rho = 0\}$, $M = \{\rho < 0\}$, $\xi \in bM$ and

$$L_\xi = \left.\frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_l}\right|_\xi$$

is positive definite.

Taylor expansion of ρ near ξ is

$$\rho(z) = \rho(\xi) + 2\,\mathfrak{Re}\varphi(\xi, z) + L_\xi(z - \xi, \bar{z} - \bar{\xi}) + \mathcal{O}(|z - \xi|^3),$$

with

$$\varphi(\xi, z) = \sum_{k=1}^{n} \left.\frac{\partial \rho}{\partial z_k}\right|_\xi (z_k - \xi_k) + \frac{1}{2}\sum_{kl=1}^{n} \left.\frac{\partial^2 \rho}{\partial z_k \partial z_l}\right|_\xi (z_k - \xi_k)(z_l - \xi_l).$$

which is holomorphic, as are all powers of φ. By the positivity of L_ξ, f has an isolated zero at ξ and all negative powers have an isolated singularity at ξ. Put $f = 1/\varphi$.

3. History: Oka-Grauert

A natural question was then whether a pseudoconvexity-type criterion was necessary for a region in $\mathbb{C}^n$ with boundary to have a similar nonextensibility property. That question was answered by Oka and Grauert over a period 1943-1958. Their methods were sheaf-theoretic and depended on the compactness of M. We will say nothing more about this.

4. History: Spencer and Kohn

Spencer's PDE and functional-analytic approach to the peak point problem led to Kohn's solution of the $\bar{\partial}$-Neumann problem in 1963–1964.

Still depends on the compactness of M, but we generalize when there are uniform structures on M given by a group action:

$$G \longrightarrow M \longrightarrow X,$$

with X compact. This is the setting in which we work.

Goal: Construct a global holomorphic function $f \neq 0$. That is an $f = f(z_1, \ldots, z_n)$ satisfying the Cauchy-Riemann equations in each variable:

$$\boxed{\frac{\partial f}{\partial \bar{z}_k} = 0 \quad (k = 1, \ldots, n).}$$

5. History: The $\bar{\partial}$-Neumann problem

Suppose M is a compact strongly pseudoconvex complex manifold and we want a holomorphic function blowing up at $z_0 \in bM$.

1) With local coordinates (z_k) solve

$$\sum \frac{\partial u}{\partial \bar{z}_k} d\bar{z}_k \overset{\text{def}}{=} \bar{\partial} u = \phi$$

in C^∞ for $\bar{\partial}\phi = 0$. This last *compatibility* condition is necessary since $\bar{\partial}^2 = 0$.

2) Take a smooth function χ with support in U_{z_0} that is identically equal 1 close to z_0 and form χf ($f = 1/\varphi$ is Levi's function) and extend by zero.

3) Since $\bar{\partial}\chi f = 0$ near z_0, $\bar{\partial}\chi f \in C^\infty(\bar{M})$. If $\bar{\partial} u = \bar{\partial}\chi f$ has smooth solution, then $\chi f - u$ is holomorphic and must blow up at z_0 since u is smooth up to the boundary. Consequently, $\chi f - u \neq 0$.

This answers a major question: Whether there are nontrivial holomorphic functions on M.

6. History: Kohn's Laplacian

Construction of solutions $u \in L^2(M) \cap \ker(\bar\partial)^\perp$ to

$$\bar\partial u = \phi$$

with $\phi \in L^2(M, \Lambda^{0,1})$, $\bar\partial\phi = 0$.

Functional analysis prefers self-adjoint operators, so for the Hilbert space adjoint $\bar\partial^*$ and seek u of the form $u = \bar\partial^* v$ satisfying

$$\bar\partial\bar\partial^* v = \phi. \tag{1}$$

Do away with the compatibility condition on ϕ (and obtain an operator related to the Dolbeault cohomology of M) by adding a term $\bar\partial^*\bar\partial v$ and define

$$\Box v \stackrel{\text{def}}{=} (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)v = \phi.$$

When $\bar\partial\phi = 0$ is true, this reduces to equation (1). Suffices to show (and this generalizes to G-manifold case) that the operator $\Box$ is Fredholm.

7. History: The $\bar{\partial}$-Neumann problem's regularity and solution

On its domain in the antiholomorphic q-forms, when $q > 0$, the operator $\Box + 1$ has the following regularity property. *Let* ζ, ζ_1 *be smooth cutoff functions for which* $\zeta_1 = 1$ *on* $\mathrm{supp}(\zeta)$ *and let* $H^s(M, \Lambda^{0,q})$ *be the integer Sobolev space of sections in* $\Lambda^{0,q}$ *over* M*. Then*

$$\Box v + v \in H^s_{\mathrm{loc}}(M, \Lambda^{0,q}) \textit{ implies } v \in H^{s+1}_{\mathrm{loc}}(M, \Lambda^{0,q})$$

and there exist constants C_s *so that*

$$\|\zeta v\|_{H^{s+1}(M)} \leq C_s \left(\|\zeta_1(\Box + 1)v\|_{H^s(M)} + \|(\Box + 1)v\|_{L^2(M)}\right)$$

uniformly in v*.*

(Estimate extended to noncompact case in Engliš, M.: Pseudolocal Estimates for $\bar{\partial}$ on General Pseudoconvex Domains, *Indiana Univ. Math. J.*, **50**, (2001) *no* 4. 1593–1607)

These inequalities imply that $(\Box + 1)^{-1} : L^2(M, \Lambda^{0,q}) \to H^1(M, \Lambda^{0,q})$ is bounded. Rellich implies $(\Box + 1)^{-1}$ is compact in $L^2(M, \Lambda^{0,q})$ implying $\dim \ker \Box < \infty$ and $\dim \mathrm{coker} \Box < \infty$ ($\Box$ is a Fredholm operator when M compact).

Also, since $\mathrm{Ker}(\Box_{p,q}) \cong H^{p,q}(M)$, we have $\dim H^{p,q}(M) < \infty$ (when M compact).

Since χf is unbounded, f^m $(m = 1, 2, \ldots)$ are linearly independent holomorphic functions in a neighborhood of z_0. Since the χf^m have compact support, $\bar{\partial}$ is injective on the vector space generated by $\{\chi f^m \mid m = 1 \ldots N\}$ (maximum modulus). It follows that for N sufficiently large,

$$Q_N = \mathrm{Im}(\square) \cap \mathrm{span}_{\mathbb{C}}\{\bar{\partial}\chi f^m \mid m = 1 \ldots N\} \neq \{0\}.$$

Thus $\bar{\partial}\bar{\partial}^* u = \phi$ can be solved for $\phi \in Q_N$. Since all the forms $\bar{\partial}\chi f^m$ are smooth, this ϕ will be smooth and so we proceed as indicated above.

9. History: The $\bar{\partial}$-Neumann problem on Covering Spaces

Here I will recap the results in Gromov, M., Henkin, G. & Shubin, M.: Holomorphic L^2 Functions on Coverings of Pseudoconvex Manifolds, *Geom. Funct. Analy.*, v. 8, *no.* 3, (1998), 552–585 ([GHS]). They obtain a similar result, that all boundary points are local peak points, when M is strongly pseudoconvex and admits a free cocompact action of a discrete group Γ by holomorphic transformations,

$$\Gamma \longrightarrow M \longrightarrow X,$$

X compact. When M is not compact, Rellich's theorem no longer holds, so the dimension of the kernel and/or cokernel of $\Box$ may be infinite-dimensional and the image of $\Box$ may be not closed.

10. Von Neumann Dimension

The *von Neumann dimension* of invariant subspaces of $L^2(\Gamma)$ is used in order to measure the kernel and cokernel of $\Box$ in this setting as well as to measure the images of $\Box$'s spectral projections. For a discrete group Γ, form the Hilbert space $L^2(\Gamma)$ with inner product $\langle \xi, \eta \rangle_{L^2(\Gamma)} = \sum_{\gamma \in \Gamma} \xi(\gamma)\bar{\eta}(\gamma)$. Then Γ acts unitarily in $L^2(\Gamma)$ by right translations R_γ, $\gamma \in \Gamma$

$$(R_\gamma \xi)(\alpha) = \xi(\alpha\gamma).$$

If an invariant subspace L is closed, then L is the image of a bounded left-convolution operator on the group:

$$L = \mathrm{Im}(L_h) \quad \text{where} \quad (L_h \xi)(\alpha) = \sum_{\gamma \in \Gamma} h(\gamma)\xi(\gamma\alpha).$$

Defining $\mathcal{B}(L^2(\Gamma))$ to be the continuous linear operators in $L^2(\Gamma)$ and

$$\mathcal{L}_\Gamma = \{L_h \mid h : \Gamma \to \mathbb{C} \text{ and } L_h \in \mathcal{B}(L^2(\Gamma))\}$$

Von Neumann's bicommutant theorem then gives that $\mathcal{L}_\Gamma$ is a von Neumann algebra.

11. Von Neumann dimension, 2

On $\mathcal{L}_\Gamma$ there is a trace defined by

$$\boxed{\mathrm{tr}_\Gamma(L_h) = h(e).}$$

and for a right-invariant subspace $L = \mathrm{Im}(L_h)$ with L_h a self-adjoint projection, we define its Γ-*dimension*

$$\dim_\Gamma(L) = \mathrm{tr}_\Gamma(L_h) = h(e).$$

Notice that since the identity in $\mathcal{B}(L^2(\Gamma))$ is convolution with δ, the characteristic function of the identity,

$$\boxed{\dim_\Gamma(L^2(\Gamma)) = \mathrm{tr}_\Gamma(L_\delta) = \delta(e) = 1}$$

though of course $\dim_\mathbb{C}(L^2(\Gamma)) = \infty$ for infinite groups.

Next, when Γ acts freely on a manifold M with compact quotient, X, one decomposes the Hilbert space $L^2(M) \cong L^2(\Gamma) \otimes L^2(X)$ and defines a trace

$$\mathrm{Tr}_\Gamma = \mathrm{tr}_\Gamma \otimes \mathrm{Tr}_{\mathcal{B}(L^2(X))}$$

on the invariant operators. This is given by choosing an orthonormal basis $(\psi_k)_k$ of $L^2(X)$ and decomposing

$$L^2(M) \cong L^2(\Gamma) \otimes L^2(X) = \bigoplus_k L^2(\Gamma) \otimes \psi_k.$$

Denoting by P_k the projection onto the k^{th} summand, a Γ-invariant operator $A : L^2(M) \to L^2(M)$ has a matrix representation $A \longrightarrow [P_k A P_l]_{kl}$ where each $A_{kl} \overset{\text{def}}{=} P_k A P_l : L^2(\Gamma) \to L^2(\Gamma)$ is bounded and invariant. The invariant trace in the invariant operators on $L^2(M)$ is then given by

$$\boxed{\mathrm{Tr}_\Gamma(A) = \sum_k \mathrm{tr}_\Gamma(A_{kk}).}$$

It is with the corresponding dimension that closed, invariant subspaces of $L^2(M)$ are measured. In [GHS], it is shown (reducing to Rellich on X) that a variant of Kohn's inequality

$$\|u\|_{H^1(M)} \leq C(\|\Box u\|_{L^2(M)} + \|u\|_{L^2(M)}),$$

implies that spectral projections $P = \int_0^\epsilon dE_\lambda$ are finite: $\mathrm{Tr}_G(P) < \infty$ thus, $\Box$ is Γ-Fredholm.

Hodge theory gives that $\dim_\Gamma \ker \Box = \dim_\Gamma L^2 \bar{H}^{p,q}(M) < \infty$ *i.e.* the reduced Dolbeault cohomology groups $L^2 \bar{H}^{p,q}(M)$ are finite Γ-dimensional.

Form the closed, invariant subspace

$$L_N = L^2(\Gamma) \otimes \mathrm{span}_{\mathbb{C}} \{\bar{\partial}\chi f, \bar{\partial}\chi f^2, \ldots, \bar{\partial}\chi f^N\} \cong L^2(\Gamma) \otimes \mathbb{C}^N$$

Because $\bar{\partial}$ is injective on functions with compact support (maximum modulus), it is on

$$\mathrm{span}_{\mathbb{C}}\{\chi f^m \mid m = 1, 2, \ldots, N\}$$

$\dim_\Gamma(L_N) = N$ and so since $\Box$ is Γ-Fredholm, $\mathrm{Im}(\Box) \cap L_N = Q \neq \{0\}$ for N sufficiently large.

Subsequently there exist closed, invariant nonempty subspaces $Q \subset \mathrm{Im}(\Box) \cap L_N$. Picking a form $\phi \neq 0$ in this Q, one sees that it is smooth so $\Box u = \phi$ is solvable and the rest of the argument is as previously described.

15. Unimodular Lie groups' von Neumann algebras

If G is unimodular Lie, then the trace on a right-invariant operator of the form $L_h^* L_h \in \mathcal{L}_G$ is uniquely determined by

$$\boxed{\mathrm{tr}_G(L_h^* L_h) = \int_G |h(t)|^2 dt.}$$

Example: If L_h is a self-adjoint projection and $h \in L^2(G)$,

$$\boxed{\mathrm{tr}_G(L_h) = \mathrm{tr}_G(L_h^* L_h) = \int_G |h(t)|^2 dt = \int_G \bar{h}(t) h(t) = \int_G h(t^{-1}) h(t) dt = h(e),}$$

since $\int_G h(t^{-1}) h(ts) dt = h(s)$. (*N.B.* $h(e)$ makes sense by Young's inequality!)

16. A Deeper Example on $G = \mathbb{R}$.

Example: If $G = \mathbb{R}$, we have ordinary convolution and the Fourier transform tells the whole story. An idempotent

$$h * h = h \quad \text{satisfies} \quad \hat{h}^2 = \hat{h}$$

and so $\hat{h}(s)$ has values in $\{0, 1\}$. The traces of these projections are then given by Plancherel:

$$\mathrm{tr}_G(h * \cdot\ ) = \mathrm{tr}_G(L_h^* L_h) = \int_G |h(t)|^2 dt = \int_{\hat{G}} |\hat{h}(s)|^2 ds = \mathrm{meas}(\mathrm{supp}(\hat{h})).$$

Let $h_N = \check{\chi}_{[-N,N]}$. Then the h_N are a family of self-adjoint projections with analytic range and

$$\dim \mathrm{Im}(h_N * \cdot) \nearrow \infty. \tag{2}$$

The ranges satisfy

$$\|u\|_{H^{s+1}(\mathbb{R})} \leq N \|u\|_{H^s(\mathbb{R})} \quad (h_N * u = u).$$

We will use functions analogous to these to mollify.

17. The Paley-Wiener theorem and its implications, 1

The Paley-Wiener theorem also comes into play in that no function $u \neq 0$ with compact support can be in a closed, invariant subspace $L \subset L^2(\mathbb{R})$ of finite G-dimension. This is because $\hat{u}$ is then analytic and so must have isolated zeros. But if $L = \mathrm{Im}(h * \cdot\ )$, $\hat{h}\hat{u} = \hat{u}$, so $\hat{h} = 1$, a.e. Thus

$$\dim_G L = \int_{\mathbb{R}} 1\ dt = \infty.$$

Note that $\dim_G L^2(G) = \infty$ for a continuous group.

18. The Paley-Wiener theorem and its implications, 2

The previous result generalizes: (Arnal, D.; Ludwig, J.: Q.U.P. and Paley-Wiener Properties of Unimodular, Especially Nilpotent, Lie Groups, *Proceedings of the AMS*, **125**, *no* 4, April 1997, 1071-1080)

Let G be unimodular Lie and $u \in L^2_c(G)$. Further, let L_h be an invariant self-adjoint projection in $L^2(G)$ for which $L_h u = u$. Then $h \notin L^2(G)$.

Idea of proof: Let χ be a cutoff function equal one in a neighborhood of support of u. Then with u_t the right-translate of u by $t \in G$,

$$L_h u_t = u_t = \chi L_h u_t,$$

for $t \in G$ sufficiently near the identity. The operator

$$(\chi L_h u)(t) = \chi(t) \int_G ds\ h(ts^{-1}) u(s)$$

has its kernel $K(t,s) = \chi(t) h(ts^{-1}) \in L^2(G \times G)$ if $h \in L^2(G)$, so is Hilbert-Schmidt and thus it is a compact operator. But this is impossible since we can find infinitely many $t \in G$ for which the translates u_t are linearly independent. Since χL_h has eigenvalue one on all the u_t we conclude $h \notin L^2(G)$. In particular,

$$\mathrm{tr}_G L = \infty \text{ if } L \text{ is closed, invariant, and contains a } u \in L^2_c(G).$$

19. The G-Fredholm property of $\Box$

In [GHS] it was shown that $\Box$ is Γ-Fredholm. The first result of the present work is an adaptation of this theorem to the situation in which the discrete group Γ is replaced by a unimodular Lie group G. Similarly, the G-Fredholm property of $\Box$ implies that the reduced Dolbeault cohomology of M is finite G-dimensional.

Theorem 19.1. *Let G be a unimodular Lie group and $G \to \bar{M} \to X$ be a principal G-bundle. Assume further that the total space M is a strongly pseudoconvex complex manifold on which G acts by holomorphic transformations and that X is compact. Then, for $q > 0$, the operator $\Box$ in $\Lambda^{p,q}(M)$ is G-Fredholm.*

20. The G-Fredholm property of the $\bar\partial$-Neumann problem

Recall the regularity theorem of Folland, Kohn, Nirenberg, Engliš:

$$\text{If } \Box v + v \in H^s_{\text{loc}}(M, \Lambda^{0,q}), \text{ then } v \in H^{s+1}_{\text{loc}}(M, \Lambda^{0,q})$$

and there exist constants C_s so that

$$\|\zeta v\|_{H^{s+1}(M)} \le C_s \left(\|\zeta_1(\Box + 1)v\|_{H^s(M)} + \|(\Box + 1)v\|_{L^2(M)}\right)$$

uniformly in v.

Thus spectral subspaces of the Laplacian are smooth:

Corollary 20.1. *Let $q > 0$ and $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of the Laplacian in $L^2(M, \Lambda^{p,q})$. If $\delta > 0$ and $P = \int_0^\delta dE_\lambda$ then $\text{im}(P) \subset C^\infty(\bar{M}, \Lambda^{p,q})$.*

Proof. Let $u \in \text{im}(P)$. Applying the theorem with $s = 0$, we have $\text{im}(P) \subset H^1_{\text{loc}}(M, \Lambda^{p,q})$. Now assume $u \in \text{im}(P) \subset H^{s-1}_{\text{loc}}(\bar{M}, \Lambda^{p,q})$. Then $(\Box + 1)u = (\Box + 1)Pu = P(\Box + 1)u \in H^{s-1}_{\text{loc}}(\bar{M}, \Lambda^{p,q})$. We conclude that $u \in H^s_{\text{loc}}(\bar{M}, \Lambda^{p,q})$ and so $\text{im}(P) \subset H^s_{\text{loc}}(\bar{M}, \Lambda^{p,q})$. □

21. The G-Fredholm property of the $\bar{\partial}$-Neumann problem, 2

We have not only C^∞ but H^∞ smoothness:

Theorem 21.1. *Let $u \in C^\infty(M, \Lambda^{0,q})$ be in the domain of the Laplacian. Then there exist constants $C_s > 0$ so that*

$$\|u\|_{H^{s+1}(M)} \leq C_s \left(\|\Box u\|_{H^s(M)} + \|u\|_{L^2(M)} \right)$$

uniformly in u.

Proof. This requires substantial modifications of Folland and Kohn's *a priori* estimates. □

Corollary 21.2. *Let $P = \int_0^\epsilon dE_\lambda$ be a spectral projection of $\Box = \int_0^\infty \lambda dE_\lambda$. Then $L^2(M) \to H^N(M)$ for any $N \in \mathbb{Z}$.*

Since $P = P^* = P^2$, spectral projections of the Laplacian $P : H^{-N}(\bar{M}) \to H^N(M)$ for any integer N. Therefore

$$(Pu)(\mathbf{x}) = \int_M K_P(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y}$$

has smooth Schwartz kernel $K_P \in C^\infty(\bar{M} \times \bar{M})$. "Proof": For N sufficiently large, we see that $P\delta = K_P \in H^N$.

22. The G-Fredholm property of the $\bar\partial$-Neumann problem, 3

Theorem 22.1. *Let* $P \in \mathcal{B}(L^2(M))^G$. *Then* $\mathrm{Tr}_G(P^*P) = \int_{\frac{M\times M}{G}} |K_P|^2$.

Proof. Since P is invariant, and its kernel $K_P \in C^\infty(\bar M \times \bar M)$, $K_P(\mathbf{x}t, \mathbf{y}t) = K_P(\mathbf{x}, \mathbf{y})$. Let $(\psi_k)_k$ be an orthonormal basis for $L^2(X)$. In the decomposition $L^2(M) \cong \bigoplus_k L^2(G) \otimes \psi_k$, P has a matrix representation $P \leftrightarrow [L_{h_{kl}}]_{kl}$. Thus,

$$\mathrm{Tr}_G(P^*P) = \textstyle\sum_{kl} \|h_{kl}\|^2_{L^2(G)}.$$

$$\mathrm{Tr}_G(P^*P) = \sum_l \mathrm{tr}_G((P^*P)_{ll}) = \sum_l \mathrm{tr}_G\left(\sum_k P^*_{kl}P_{kl}\right) = \sum_{kl} \mathrm{tr}_G(L^*_{h_{kl}}L_{h_{kl}}) = \sum_{kl} \|h_{kl}\|^2_{L^2(G)}.$$

23. The G-Fredholm property of the $\bar\partial$-Neumann problem, 4

Choose a section $x : X \to M$ and represent points $\mathbf{x} \in M$, $\mathbf{x} \to (t, x) \in G \times X$. Whenever $P \in \mathcal{B}(L^2(M))^G$ and $K_P \in L^2_{\text{loc}}(M \times M)$, we can write

$$K_P(\mathbf{x}, \mathbf{y}) \longrightarrow K_P(t, x; s, y) \stackrel{\text{def}}{=} \kappa(ts^{-1}; x, y), \quad s, t \in G,\ x, y \in X$$

with $\kappa \in L^2_{\text{loc}}(G \times X \times X)$.

Now, except on a set of measure zero, we have a description of P

$$(Pu)(\mathbf{x}) = \int_M K_P(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\mathbf{y} = (Pu)(t, x) = \int_{G \times X} dsdy\ \kappa(s; x, y) u(st, y).$$

The distributional kernels h_{ij} can be recovered from κ by projecting into the summands in

$$L^2(M) \cong \bigoplus_l (L^2(G) \otimes \psi_l),$$

$$h_{ij} = \int_{X\times X} dxdy\ \kappa(\ \cdot\ ; x, y)\psi_j(y)\overline{\psi}_i(x).$$

Compute the norm of κ in $L^2(G \times X \times X)$. Since $(\psi_j)_j$ is an orthonormal basis for $L^2(X)$, $(\overline{\psi}_i \otimes \psi_j)_{ij}$ an orthonormal basis for $L^2(X \times X)$. By construction, h_{ij} is equal the ij^{th} Fourier coefficient of κ with respect to the decomposition $L^2(G \times X \times X) \cong \bigoplus_{ij}(L^2(G) \otimes \psi_i \otimes \psi_j)$. Hence

$$\sum_{ij} \|h_{ij}\|^2_{L^2(G)} = \|\kappa\|^2_{L^2(G\times X\times X)}.$$

Thus

$$\boxed{\mathrm{Tr}_G(P^*P) = \|\kappa\|^2_{L^2(G\times X\times X)} = \int_{\frac{M\times M}{G}} |K_P(\mathbf{x}, \mathbf{y})|^2\ \frac{d\mathbf{x}d\mathbf{y}}{dt}.}$$

□

25. Finiteness of the G-trace on $\Box$'s spectral projections, 1

Lemma 25.1. *If $P \in \mathcal{B}(L^2(M))^G$ is a self-adjoint, invariant projection so that* $\mathrm{im}(P) \subset C^\infty(M)$*, then* $K_P \in L^2(\frac{M\times M}{G})$.

Proof. Fix $\mathbf{x} \in M$. If $P : L^2(M) \to C^\infty(M)$, the closed graph theorem applied to P implies

$$L^2(M) \ni u \longmapsto (Pu)(\mathbf{x}) \in \mathbb{C}$$

is a bounded linear functional. The Riesz representation theorem then gives that there exists a function $h_{\mathbf{x}} \in L^2(M)$ so that

$$(Pu)(\mathbf{x}) = \langle h_{\mathbf{x}}, u\rangle = \int_M K_P(\mathbf{x},\mathbf{y})u(\mathbf{y})d\mathbf{y} \quad u \in L^2(M).$$

So,

$$\phi(\mathbf{x}) \stackrel{\text{def}}{=} \int_M |K_P(\mathbf{x},\mathbf{y})|^2 d\mathbf{y} < \infty \text{ for any } \mathbf{x} \in M.$$

26. Finiteness of the G-trace on $\Box$'s spectral projections, 2

The function ϕ is constant on orbits since the measure on M is invariant;

$$\phi(\mathbf{x}t) = \int_M |K_P(\mathbf{x}t, \mathbf{y})|^2 d\mathbf{y} = \int_M |K_P(\mathbf{x}, \mathbf{y}t^{-1})|^2 d\mathbf{y} = \int_M |K_P(\mathbf{x}, \mathbf{y})|^2 d\mathbf{y} = \phi(\mathbf{x}).$$

Thus ϕ descends to a function on $M/G = X$. Since the map from M to $C_c^{-\infty}(M)$ defined by $\mathbf{y} \mapsto \delta_{\mathbf{y}}$ is continuous, the composition

$$\mathbf{y} \mapsto P\delta_{\mathbf{y}} = K_P(\cdot, \mathbf{y})$$

is a continuous function $M \to L^2(M)$. We may conclude that $\phi : X \to \mathbb{R}_+$ is continuous. Denote by $\frac{d\mathbf{x}}{dt}$ the quotient measure on X. The compactness of X together with continuity of ϕ imply that $\int_X \phi(\mathbf{x}) \frac{d\mathbf{x}}{dt} < \infty$. Thus we have that $K_P \in L^2(\frac{M \times M}{G})$. $\Box$

Conclude that

$$\mathrm{Tr}_G(P) = \int_{\frac{M\times M}{G}} |K_P(\mathbf{x}, \mathbf{y})|^2 \frac{d\mathbf{x}d\mathbf{y}}{dt} < \infty \text{ for any } P = \int_0^\epsilon dE_\lambda.$$

This implies that $\Box$ is G-Fredholm.

27. Closed, invariant subspaces of $L^2(M)$

If $f \in L^2(M)$, define $\langle f\rangle^\circ$ to be the vector space generated by finitely many right translates of f:

$$\langle f\rangle^\circ = \left\{ \sum_{t\in G} \alpha_t f(\cdot t) \mid \alpha : G \to \mathbb{C} \text{ has finite support} \right\}.$$

Define also

$$\langle f\rangle = \overline{\langle f\rangle^\circ}^{L^2(M)},$$

the closure of $\langle f\rangle^\circ$ in $L^2(M)$.

Example: With $G = \mathbb{R}$ and $M = G \times X$ a trivial bundle, we can write $f = f(t,x)$ and construct the minimal invariant projection onto $\langle f\rangle$:

$$\langle f\rangle = \mathrm{Im}(P), \quad (Pu)(t,x) = \int_{\mathbb{R}\times X} K_P(t,x;s,y)u(s,y)dsdy = \int_{\mathbb{R}\times X} \kappa(t-s;x,y)u(s,y)dsdy,$$

define the partial Fourier transform $u \mapsto \tilde{u}$ by $\tilde{u}(\tau,x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-it\tau}u(t,x)dt$, then $Pf = f$ iff

$$\tilde{f}(\tau,x) = \int_X \tilde{\kappa}(\tau;x,y)\tilde{f}(\tau,y)dy.$$

Note that if $Pf = f$, then for any function ϕ,

$$\phi(\tau)\tilde{f}(\tau, x) = \int_X \tilde{\kappa}(\tau; x, y)\phi(\tau)\tilde{f}(\tau, y)dy.$$

This is equivalent to translation-invariance. Assuming without loss of generality that for each τ, $\|\tilde{f}(\tau, \cdot)\| \in \{0, 1\}$, $\tilde{\kappa}$ has a *minimal* (rank-one) form:

$$\tilde{\kappa}(\tau; x, y) = \tilde{f}(\tau, x) \otimes \overline{\tilde{f}(\tau, y)}.$$

In Fourier space, $P^2 = P$ is $\int \tilde{\kappa}(\tau; x, y)\tilde{\kappa}(\tau; y, z)dy = \tilde{\kappa}(\tau; x, z)$ and $P = P^*$ is $\tilde{\kappa}(\tau; x, y) = \overline{\tilde{\kappa}(\tau; y, x)}$, so

$$\dim_G\langle f\rangle = \int_{\hat{G}\times X\times X} |\tilde{\kappa}(\tau; x, y)|^2 d\tau dx dy = \int_{\hat{G}\times X\times X} \tilde{\kappa}(\tau; y, x)\tilde{\kappa}(\tau; x, y) d\tau dx dy = \int_{\hat{G}\times X} \tilde{\kappa}(\tau; y, y) d\tau dy$$

$$= \int_{\hat{G}\times X\times X} |\tilde{f}(\tau, y)|^2 d\tau dy = \int_{\hat{G}} \mathbf{1}\ d\tau = +\infty.$$

29. A Paley-Wiener Type Theorem

We now use the G-dimension to measure closed, invariant subspaces of $L^2(M, E)$ like $\langle \bar{\partial}\chi f \rangle$ and $\overline{\bar{\partial}\langle \chi f \rangle}$. The methods here are simple adaptations of those in Arnal & Ludwig's.

Theorem 29.1. *Let $G \to M \to X$ be as above and let $f \in L^2(M)$ have compact support. Assume further that there exists an invariant operator $P \in \mathcal{B}(L^2(M))^G$ satisfying $Pf = f$ and with kernel $\kappa \in L^2(G \times X \times X)$. Then $f = 0$, $\mu - a.e.$*

Proof. Similar to before. □

Thus, since

$$\bar{\partial}\chi f \in C_c^\infty(M),\ \dim_G \langle \bar{\partial}\chi f \rangle = \infty.$$

And choosing the power $t \in \mathbb{R}$ appropriately so that $\chi f^t \in L_c^2(M)$,

$$\dim_G \langle \chi f^t \rangle = \infty.$$

30. Crash!

Since $\Box$ is G-Fredholm, $\mathrm{Im}(\Box) \cap \langle\bar{\partial}\chi f\rangle$ has closed, invariant subspaces of arbitrarily large G-dimension. It turns out that $\bar{\partial}\langle\chi f\rangle \cap \langle\bar{\partial}\chi f\rangle$ also has closed, invariant subspaces of arbitrarily large G-dimension, so in fact,

$\mathrm{Im}(\Box) \cap \langle\bar{\partial}\chi f\rangle \cap \bar{\partial}\langle\chi f\rangle$ has closed, invariant subspaces L of arbitrarily large G-dimension.

Now for $0 \neq g \in L$ we solve

$$\Box u = g.$$

Integrating by parts or noting that $\mathrm{Im}(\overline{\partial}) \perp \mathrm{Im}(\overline{\partial}^*)$ and $g \in \overline{\mathrm{Im}(\overline{\partial})}$ we have

$$\Box u = \overline{\partial}\overline{\partial}^* u = g = \overline{\partial}\phi$$

for some $\phi \in \langle\chi f\rangle$.

Now, for smoothing projections $P = R_k \in \mathcal{L}_G$ of sufficiently large G-trace, $R_k g \neq 0$. So

$$R_k \Box u = \Box R_k u = R_k g = R_k \bar{\partial} \phi = \bar{\partial} R_k \phi.$$

Form

$$\Phi = R_k(\phi - \overline{\partial}^* u).$$

If we can show that

$$\boxed{R_k g \in R_k \langle \overline{\partial} \chi f \rangle \subset C^\infty(M, \Lambda^{0,1})}$$

and that

$$\boxed{R_k \phi \notin C^\infty(\bar{M})}$$

by showing that $C^\infty(\bar{M}) \cap \langle \chi f \rangle = \{0\}$, the regularity of $\Box$ will give that $\overline{\partial}^* u$ and thus $R_k \overline{\partial}^* u$ are smooth and so $\Phi \neq 0$.

This turns out to be true, but that is another story.