CLOSED, INVARIANT SUBSPACES OF $L^2(G)$, THEIR VON NEUMANN DIMENSIONS, AND PALEY-WIENER THEOREMS

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1. Sobolev-type estimates

Definition 1.1. *For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ in the norm $\|\cdot\|_s$ defined by*

$$\|u\|_s^2 = \int_{\mathbb{R}^n} |(1-\Delta)^{s/2} u|^2$$

where Δ is the Laplacian on $\mathbb{R}^n$.

Note that $\|\cdot\|_0 = \|\cdot\|$ is the ordinary $L^2(\mathbb{R}^n)$ norm. We will use the two notations interchangeably. By Plancherel's theorem and the fact that the Fourier transform preserves Schwartz functions,

$$\|u\|_s^2 = \int_{\mathbb{R}^n} |(1+\xi^2)^{s/2} \hat{u}(\xi)|^2.$$

Suppose we have a (necessarily closed) subspace $L \subset L^2(\mathbb{R})$ that is translation-invariant and satisfying a Sobolev-type estimate

$$\|u\|_1 \leq C\|u\|_0 \quad (u \in L). \tag{1}$$

These come up in PDE in the study of elliptic and subelliptic operators. If $\mathbb{R}$ were replaced by a compact, the estimate would imply automatically that L is finite-diminsional (by Rellich's lemma). I'll say more later.

Any closed subspace L of a Hilbert space is the image of a self-adjoint projection P_L and for it to be invariant implies that the projection has an integral kernel $K_P(x,y)$ that depends only on $x-y$:

$$L = \mathrm{Im}(P_L) \quad (P_L u)(x) = \int_{\mathbb{R}} dy\ K(x,y)u(y) = \int_{\mathbb{R}} dy\ k(x-y)u(y) = k * u(x).$$

In short, P_L is a convolution operator. The conditions that P_L be a self-adjoint projection imply, respectively, that $k(x-y) = K(x,y) = \overline{K(y,x)} = \overline{k(y-x)}$ and

$$k(x-y) = K(x,y) = \int_{\mathbb{R}} dz\ K(x,z)K(z,y) = \int_{\mathbb{R}} dz\ k(x-z)k(z-y) = k*k(x-y).$$

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Notice that such a function also satisfies $k' = k' * k = k * k'$, $k'' = k'' * k = k' * k' = k * k'', \dots$.

Just looking for any nonzero function k satisfying $k * k = k$ seems an impossible task, even without the requirement that $L = \{k * u \mid u \in L^2(\mathbb{R})\}$, but the Fourier transform's taking the convolution to ordinary pointwise multiplication makes this a snap:

$$k * k(x) = k(x) \quad \Leftrightarrow \quad \hat{k}(\xi) \cdot \hat{k}(\xi) = \hat{k}(\xi)$$

implies that $\hat{k} : \mathbb{R} \to \{0, 1\}$. So if h is the characteristic function of any measurable set S,

$$\chi_S(x) \stackrel{\text{def}}{=} \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}$$

then its inverse transform $\check{h}$ satisfies $\check{h} * \check{h} = \check{h}$.

Now supposing that L is the image of an operator of the form $u \mapsto \check{\chi}_S * u$, the estimate $\|u\|_1 \leq C\|u\|_0$ gives

$$\|\check{\chi}_S * u\|_1 \leq C\|\check{\chi}_S * u\|_0 \quad (u \in L^2(\mathbb{R}))$$

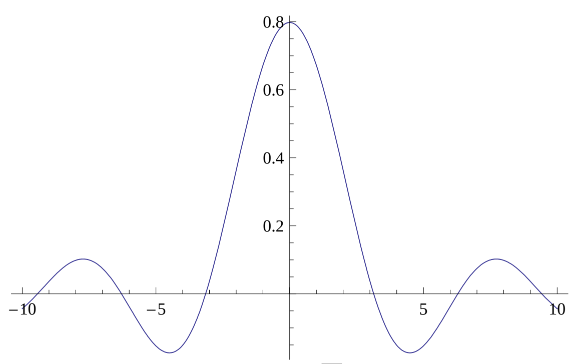
which is equivalent to

$$\int [C^2 - (1 + \xi^2)]|\chi_S(\xi)\hat{u}(\xi)|^2 d\xi \geq 0 \quad (u \in L^2(\mathbb{R})). \tag{2}$$

Since we may take functions $\hat{u} \in L^2(\mathbb{R})$ with small support wherever we please, we conclude that

$$S = \{\xi \in \mathbb{R} \mid C^2 - (1 + \xi^2) \geq 0\} = [-\sqrt{C^2 - 1}, \sqrt{C^2 - 1}].$$

Suppose $S = [-1, 1]$, then taking the inverse Fourier transform, we obtain



$$\check{\chi}_S(x) = \sqrt{\frac{2}{\pi}} \frac{\sin(x)}{x}$$

Notice that the invariance, closure and estimate give something for free: The only requirement we had to satisfy in Equation (2) was that $C^2 \geq 1 + \xi^2$. Raising both sides to whichever power we please does not change anything, so the estimate (1) is equivalent to

$$\|u\|_s \leq C^s \|u\|_0 \quad (s \in \mathbb{R}_+,\ u \in L).$$

Note that L consists of smooth functions in this case. We could have concluded this from the Paley-Wiener theorem which asserts (among other things) that a function with compact support has a real-analytic Fourier transform.

There is nothing special about $\mathbb{R}$ in the above other than it be an abelian Lie group, so this generalizes. Suppose now that G is a group that has a biinvariant elliptic Laplacian, Δ_G. This happens exactly when G is compact and semisimple.

The Fourier transform is more complicated in that case. It is called the Peter-Weyl theorem, which we state: *If G is a compact group, then the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of G is dense in $L^2(G)$.*
For our spaces L, what will happen here is that the matrix elements are all analytic and the inequality will guarantee that only finitely many will occur in the expansion of any element of L. Yet another approach is a simple application of Rellich's lemma. [1]

This will work fine to describe the analogues of our spaces L, but let's avoid all this machinery for now and restrict to our semisimple case and method.

The estimate in Equation (1) makes perfect sense once one chooses an invariant (Haar) measure and an invariant Laplacian (called the Casimir) on the group. Let us similarly assume that L is invariant under right-translations of the group:

$$u \in L \quad \Rightarrow \quad (R_t u)(x) = u(xt) \in L \quad (t \in G).$$

Since any left-convolution

$$(L_h u)(x) = \int_G dt\ h(t) u(t^{-1}x)$$

will commute with right-translations, we seek a function h on G for which L_h is a self-adjoint projection in $L^2(G)$ having the desired image,

$$L = \{u \in L^2(G) \mid \|u\|_1 \leq C\|u\|_0\}.$$

If $L_h = L_h^* = L_h^2$, abbreviating $\partial = \sqrt{1+\Delta_G}$, we have $L_h = L_h L_h$, $L_{\partial h} = L_{\partial h} L_h = L_h L_{\partial h}$ we have $\partial^2 h = L_{\partial h} L_{\partial h}, \ldots,\ L_{\partial^{(k)} h} = L_{\partial h}^k$. If we assume that $\mathrm{Im}(L_h) = L$, then for all $u \in L^2(G)$ the following holds by assumption:

$$\|\partial L_h u\|_{L^2(G)} = \|L_{\partial h} u\|_{L^2(G)} = \|L_h u\|_{H^1(G)} \leq C \|L_h u\|_{L^2(G)} \quad (u \in L^2(G)).$$

But then for all $u \in L^2(G)$

$$\|L_h u\|_{H^2(G)} = \|L_{\partial^2 h} u\|_{L^2(G)} = \|L_{\partial h}^2 u\|_{L^2(G)} \leq C\|L_{\partial h} u\|_{L^2(G)} \leq C^2 \|u\|_{L^2(G)}.$$

[1] The key example here is to take G equal the circle group and see that the estimate forces Fourier coefficients a_ξ to vanish for $|\xi| \geq \sqrt{C^2-1}$, exactly as above for $G = \mathbb{R}$.

The first inequality above follows from the assumption because $L_{\partial h}u \in L^2(G)$. So we deduce that $\|L_h u\|_{H^2(G)} \leq C^2\|u\|_{L^2(G)}$. Iterating this procedure we have that $L_h u \in H^\infty(G)$ and so is smooth for all $u \in L^2(G)$. Therefore h is smooth. Note that the growth of the constants is identical to previously: $\|L_h u\|_k \leq C^k\|u\|_0$.

Groups that are neither compact nor commutative are significantly tougher to handle; that there is no biinvariant Laplacian is the main difficulty. The only way out is to throw functional analysis at the problem.

Lemma 1.2. *Let $L_h \in \mathcal{B}(L^2(G))^G$ be an invariant self-adjoint projection such that* $\mathrm{Im}(L_h) \subset C^\infty(G)$*. Then $h \in C^\infty(G)$.*

Proof. Closed graph theorem and some facts about Fréchet spaces and their duals. □

Lemma 1.3. *If $L_h : L^2(G) \to C^\infty(G)$ is a self-adjoint projection, then $h \in L^2(G)$.*

Proof. More Fréchet topology and a Riesz representation. □

2. Von Neumann dimension

Let L be a closed subspace of $L^2(G)$ invariant under right-translations of the group and $L = \mathrm{Im}(L_h)$ for some distribution h on G. Then define the G-dimension of L by

$$\dim_G(L) = \|h\|^2.$$

It is true that this definition of a dimension has all the properties one would expect. For example $\dim_G(L_1 \oplus L_2) = \dim_G(L_1) + \dim_G(L_2)$ and so on.

For our examples from the previous section, with $L = \{u \in L^2(\mathbb{R}) \mid \|u\|_1 \leq C\|u\|_0\}$, we got $L = \mathrm{Im}(\check{\chi}_S * \cdot\ )$ with $S = [-\sqrt{C^2-1}, \sqrt{C^2-1}]$. Thus

$$\dim_G(L) = 2\sqrt{C^2-1},$$

by Plancherel's theorem.

Notice that the complex dimension of $L \cong L^2([-\sqrt{C^2-1}, \sqrt{C^2-1}])$ is infinite for $C > 1$.

Lemma 2.1. *If $P : L^2(G) \to C^\infty \cap L^2(G)$ is an invariant self-adjoint projection, then $P = L_h$ with $h \in C^\infty \cap L^2(G)$ and so* $\dim_G P < \infty$.

Proof. Combine the previous two lemmas with the formula $\dim_G(\mathrm{Im}(L_h)) = \|h\|^2$. □

Notice we have to assume more than the estimate (1).

3. Paley-Wiener Theorems

We will only discuss a simple consequence of the Paley-Wiener theorem. If u is a distribution on $\mathbb{R}^n$ with compact support, then $\hat{u}$, its Fourier transform, is analytic. Such $\hat{u}$ can then have only isolated zeros. We may conclude that if $\chi_S \cdot \hat{u} = \hat{u}$, then $\chi_S = 1$ almost everywhere. Thus the G-dimension of *any* closed, invariant space containing *any* u with compact support is infinite! In fact, the linear span of the translates of u are dense in $L^2(\mathbb{R}^n)$. This density property is false already for the group $\{(e^{i\theta}, t) \mid \theta, t \in \mathbb{R}\}$ with the operation $(\cdot, +)$.

It turns out though, that he previous assertion is true in great generality [AL]. We sketch the result and its proof.

Theorem 3.1. *Let G be a noncompact, connected, unimodular, Lie group. Furthermore let $f \in L^2(G)$ have compact support. Assume further that there exists an invariant operator L_h with nonzero $h \in L^2(G)$ satisfying $L_h f = f$. Then $f = 0$ a.e.*

Proof. The assumptions guarantee the existence of a sequence $(x_k)_k \subset G$ for which $S = \cup_k(\mathrm{supp}(f) \cdot x_k)$ has finite measure and the functions $R_{x_k} f$ are linearly independent. One sees that all of the functions $R_{x_k} f$ are eigenfunctions of the operator $\chi_S L_h$, with eigenvalue one. Thus it is not a compact operator. We compute the Hilbert-Schmidt norm of the operator $\chi_S L_h$. Since

$$(\chi_S L_h u)(x) = \chi_S(x) \int_G dt\ h(t) u(t^{-1}x) = \chi_S(x) \int_G dt\ h(xt^{-1}) u(t)$$

the kernel of $\chi_S L_h$ is $K(x,t) = \chi_S(x) h(xt^{-1})$. Thus

$$\|\chi_S L_h\|_{HS}^2 = \int_{G\times G} dxdt\ |\chi_S(x) h(xt^{-1})|^2 = \|h\|^2 \int_G dx\ |\chi_S(x)|^2.$$

Since $\int_G dx\ |\chi_S(x)|^2 = meas(S)$, we must have $\|h\|^2 = \infty$ otherwise $\chi_S L_h$ would be compact. □

References

[AL] Arnal, D.; Ludwig, J.: Q.U.P. and Paley-Wiener Properties of Unimodular, Especially Nilpotent, Lie Groups, *Proceedings of the AMS*, **125**, *no* 4, April 1997, 1071-1080

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