RESEARCH STATEMENT

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1. Overview

My research focuses on the existence theory of invariant, linear partial differential equations on G-manifolds M with compact quotient M/G.

More specifically, Folland, Kohn, and Nirenberg's solution of the $\bar{\partial}$-Neumann problem, as presented in [FK], is valid for compact, strongly pseudoconvex complex manifolds and provides an excellent model for general subelliptic problems.

My research program involves the generalization of as many results as possible from [FK] and from related works (*e.g.* [E, GHS]) to the setting in which M is an analogous G-manifold, with G a unimodular Lie group acting freely in M by holomorphic transformations:

$$G \longrightarrow M \longrightarrow X.$$

This work falls mainly into two branches at present:

1.1. G-Fredholm Property.

In [P1] the G-Fredholm property of Kohn's Laplacian was established for strongly pseudoconvex G-manifolds as described above. Several ramifications that will be investigated in the near future are the analogous property of the boundary Laplacian $\Box_b$, leading eventually to inquiries regarding the holomorphic extension of boundary values on M. This work probably has its farthest-reaching consequences in its relation to general subelliptic problems as in [Ep].

1.2. The Levi Problem.

The solution to the Levi problem on such M as described in [P2] brings up two avenues of further research. The more widely applicable one, in our view, lies in its use of harmonic analysis to generate large, smooth, invariant, closed subspaces of $L^2(M)$ for which the G-Fredholm property of [P1] can be exploited. The other route suggested here is toward a number of differential-geometric questions that are quite natural to the geometry of the Grauert tubes of Lie groups and their extensions, particularly as constructed in [HHK]. The main property worth further investigation here is that of amenability as defined in [P2].

The techniques used include hard analysis, functional analysis, von Neumann algebras, differential geometry, and noncommutative harmonic analysis.

2. Introduction

2.1. Levi's problem.

Let the space of holomorphic functions on Ω be denoted $\mathcal{O}(\Omega)$. If $x \in b\Omega$ and there is a function $f \in \mathcal{O}(\Omega)$ unbounded in any neighborhood of x and bounded in its complement, then x is called a *peak point* of $\mathcal{O}(\Omega)$. A region Ω for which every point of the boundary is a peak point for $\mathcal{O}(\Omega)$ is called a *domain of holomorphy.*

It has long been a goal in the theory of several complex variables to understand, geometrically, the smooth regions $\Omega \subset \mathbb{C}^n$ that are domains of holomorphy. In 1910, Eugenio E Levi proposed, in [L, L1, L2], a geometric property related to the usual idea of convexity:

Definition 2.1. *Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be smooth and $\Omega \subset \mathbb{C}^n$ so that $\Omega = \{z \in \mathbb{C}^n \mid \rho(z) < 0\}$ and $d\rho \neq 0$ on $b\Omega$. We say that Ω is strongly pseudoconvex if at each point $x \in b\Omega$, the form*

$$L_x = \left.\frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_l}\right|_x$$

is positive definite.

One aspect of our work is to construct holomorphic functions on such regions as Ω and on manifolds. Some will have peak points and others will have weaker singularities.

If we are discussing complex manifolds, then we modify the definition of pseudoconvexity as follows. We assume that the complex manifold M is a subset of a slightly bigger complex manifold $\widetilde{M}$ of the same dimension so that $\bar{M} \subset \widetilde{M}$. In this case, if $\rho : \widetilde{M} \to \mathbb{R}$, the definition above makes sense as does the idea of prolonging a holomorphic function beyond bM.

Consider the Taylor expansion of ρ near $x \in bM$,

$$\rho(z) = \rho(x) + 2\,\mathfrak{Re}\varphi(z,x) + L_x(z-x, \bar{z}-\bar{x}) + \mathcal{O}(|z-x|^3),$$

with

$$\varphi(z,x) = \sum_{k=1}^{n} \left.\frac{\partial \rho}{\partial z_k}\right|_x (z_k - x_k) + \frac{1}{2}\sum_{kl=1}^{n} \left.\frac{\partial^2 \rho}{\partial z_k \partial z_l}\right|_x (z_k - x_k)(z_l - x_l).$$

The function $\varphi(\cdot, x)$ is obviously holomorphic in a neighborhood of x. Given the positivity of L_x, it follows that $\varphi|_{\bar{M}}$ has an isolated zero at x, and that the logarithm of φ is holomorphic in M near x. In 1958, Grauert showed in [G] (using sheaves) that if M is strongly pseudoconvex, compact, and $bM \neq \emptyset$, then every point $x \in bM$ is a peak point for $\mathcal{O}(M)$. It follows in particular that the space $\mathcal{O}(M)$ is infinite-dimensional.

In the 1950s, Spencer's PDE and functional-analytic approach to the peak point problem led to Kohn's solution of the $\bar{\partial}$-Neumann problem in 1963–1964. We will work with methods similar to theirs.

Spencer's method is to solve

$$\bar{\partial} u = \phi$$

in C^∞ for $\bar{\partial}\phi = 0$. If this can be done, then take a smooth function χ with support in a neighborhood of x, that is identically equal 1 close to x, and form χf, with $f = \log \varphi$, and extend by zero to all of M.

Since f is holomorphic (*i.e.* $\bar{\partial}\chi f = 0$) near x, we have that $\bar{\partial}\chi f$ has a smooth extension to $\bar{M}$. If $\bar{\partial} u = \bar{\partial}\chi f$ has smooth solution, then $\chi f - u$ is holomorphic and must blow up at x since u is smooth up to the boundary. In particular, $\chi f - u \neq 0$, so this also indicates that there exist nontrivial holomorphic functions on M. To solve $\bar{\partial} u = \phi$, with $\phi \in L^2(M, \Lambda^{0,1})$, $\bar{\partial}\phi = 0$ we adjust the problem somewhat. For technical reasons, we instead solve

$$\Box v \overset{\text{def}}{=} (\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})v = \phi$$

with $\bar{\partial}^*$ the Hilbert space adjoint of $\bar{\partial}$ and $v = \bar{\partial}^* u$. To solve $\Box v = \phi$, we will show that it is Fredholm in various senses. These have in common the feature that the kernel of the operator be finite-dimensional, and that the image of the operator contain a closed subspace of $L^2(M)$ of finite codimension, though the idea of dimension will change according to M, as will be described below.

3. Fredholm properties of $\Box$

3.1. **Background.**

The Kohn Laplacian $\Box$ has the following regularity property.

Theorem 1 [FK] *Let ζ, ζ_1 be smooth cutoff functions for which $\zeta_1 = 1$ on $\operatorname{supp}(\zeta)$ and let $H^s(M, \Lambda^{0,q})$ be the integer Sobolev space of sections in $\Lambda^{0,q}$ over M. Then there exist constants C_s so that*

$$\|\zeta v\|_{H^{s+1}(M)} \leq C_s \left(\|\zeta_1(\Box + 1)v\|_{H^s(M)} + \|(\Box + 1)v\|_{L^2(M)} \right)$$

uniformly in v.

This theorem is valid also in the noncompact case, even without the assumption of uniformity of structures [E]. Note that there are examples of G-bundles on which there are no nontrivial square-integrable holomorphic functions, [GHS].

In each of the following cases of M, Theorem 1 (and/or a variant) is used to demonstrate the applicable Fredholm property.

Compact case. If M is compact, Theorem 1 implies that

$$(\Box + 1)^{-1} : L^2(M, \Lambda^{0,q}) \to H^1(M, \Lambda^{0,q})$$

is bounded. Rellich implies $(\Box + 1)^{-1}$ is compact in $L^2(M, \Lambda^{0,q})$ implying $\dim_{\mathbb{C}} \ker \Box < \infty$ and $\dim_{\mathbb{C}} \mathrm{coker}\Box < \infty$, thus $\Box$ is a Fredholm operator when M compact.

Covering spaces. When M is not compact, Rellich's theorem no longer holds, so the dimension of the kernel and/or cokernel of $\Box$ may be infinite-dimensional and the image of $\Box$ may be not closed. But if M is a regular covering,

$$\Gamma \longrightarrow M \longrightarrow X$$

(Γ discrete, X compact), then there is a trace Tr_Γ on the von Neumann algebra of Γ-invariant operators in $L^2(M)$ with which we can measure invariant projections and thus their images. In [GHS], it is shown (by reducing to Rellich on X) that the estimate

$$\|u\|_1 \lesssim \|\Box u\|_0 + \|u\|_0 \qquad (u \in C^\infty(M, \Lambda^{0,1}) \cap \mathrm{Dom}(\Box)) \tag{1}$$

implies that spectral projections $P_\delta = \int_0^\delta dE_\lambda$ of $\Box$ are finite: $\mathrm{Tr}_\Gamma(P_\Delta) < \infty$. This implies that $\dim_\Gamma \ker \Box < \infty$ and $\mathrm{im}P_\delta \subset \mathrm{im}\Box$ is finite Γ-dimensional, thus $\Box$ is Γ-Fredholm. The estimate (1) would have sufficed in the compact case, of course.

3.2. **Completed research.** In [P1], the covering space is generalized to a G-bundle with G a unimodular Lie group.

Theorem 2 *Assume that G is a unimodular Lie group and $G \to M \to X$ a principal G-bundle. Assume further that the total space M is a strongly pseudoconvex complex manifold on which G acts by holomorphic transformations and that X is compact. Then, for $q > 0$, the operator $\Box$ in $\Lambda^{p,q}(M)$ is G-Fredholm.*

The method of proof is first to show that Theorem 1 implies that the spectral projections of the Laplacian are smoothing: $P_\delta : L^2(M) \to C^\infty(M)$.

In [P1] a new local *a priori* estimate was obtained:

$$\|\zeta_1 u\|_{s+1} \lesssim \|\zeta \Box u\|_s + \|\zeta u\|_0 \quad (u \in C^\infty(M, \Lambda^{p,q}), q > 0)$$

which can be glued to obtain $\|u\|_{s+1} \lesssim \|\Box u\|_s + \|u\|_0$ assuming $q > 0$ and u smooth in the domain of $\Box$.

These estimates together imply that $P_\delta : L^2(M) \to H^\infty(M)$ which gives that the Schwartz kernel of P_δ is smooth. It follows that $\mathrm{Tr}_G(P_\delta) < \infty$. This is the same as saying the Laplacian is G-Fredholm.

There are numerous corollaries in this paper relating the G-Fredholm property to the reduced Dolbeault cohomology of M and (using the Hodge decomposition) to the (non self-adjoint) operator $\bar\partial$.

3.3. **Future directions.** Deepening the techniques in [P1] regarding the properties of the spectral projections of $\Box$, we will be able to construct a family of approximate (bounded) Neumann operators N for $\Box$. These will be used to correct the later claims of [E] regarding the Bergman kernel and the existence of nontrivial global holomorphic functions on M without the machinery (or limitations) of the next section. These holomorphic functions will probably not demonstrably have peak behavior in bM and so will be irrelevant to the Levi problem.

The methods in [P1] can easily be adapted to obtain the G-Fredholm property for the boundary Laplacian $\Box_b$. Analogously we will be able to construct bounded approximate inverses (this time called G_b in [FK]). These would lead later to a study of the holomorphic extensibility of boundary values.

Ultimately this property should hold for general invariant subelliptic problems as suggested in [Ep].

4. The Levi Problem

4.1. **Background.** An important part of solving the Levi problem, whether M is compact, a covering space, or the total space of a G-bundle, is in establishing that $\Box$ is Fredholm in the appropriate sense. The application of the Fredholm property is described in the following.

Compact case. Let $f = \log \varphi$ and let $\chi \in C_c^\infty(M)$, identically equal 1 near $x \in bM$. Since f is unbounded, the powers of f are linearly independent and since $\bar\partial$ is an injection on functions with small support, the space

$$L_N = \operatorname{span}\{\bar\partial \chi f^m \mid m = 1, \dots, N\}$$

has $\dim_{\mathbb{C}} L_N = N$. The Fredholm property implies that, for N sufficiently large, $\operatorname{im}\Box \cap L_N \neq \{0\}$. Note that the $\bar\partial \chi f^m$ have smooth extensions to $\bar M$. Choosing an element

$$\phi = \bar\partial \sum_{k=1}^{N} \alpha_k \chi f^k \overset{\text{def}}{=} \bar\partial \psi$$

in the intersection, we solve $\bar\partial u = \bar\partial \psi$, $u \perp \ker \bar\partial$. Since $\phi \in C^\infty(\bar M)$, regularity gives that $u \in C^\infty(\bar M)$. Forming $\psi - u$, we obtain a holomorphic function blowing up at $x \in bM$.

Covering spaces. There is no change in the argument except in the definition of the spaces L_N. As we need Γ-invariant subspaces of $L^2(M)$, we instead generate the space of linear combinations of translates of the $\bar\partial \chi f^m$. That is to say that we take all convolutions of these forms:

$$\begin{aligned} L_N = \{\sum_{\gamma \in \Gamma} \sum_{m=1}^{N} \alpha_{m,\gamma} (\bar\partial \chi f^m)(\cdot\, \gamma^{-1}) \mid \sum_{m,\gamma} |\alpha_{m,\gamma}|^2 < \infty\} \\ \cong L^2(\Gamma) \otimes \operatorname{span}_{\mathbb{C}} \{\bar\partial \chi f, \bar\partial \chi f^2, \dots, \bar\partial \chi f^N\} \cong L^2(\Gamma) \otimes \mathbb{C}^N. \end{aligned}$$

The Γ-dimension of L_N then is N and the rest of the argument is identical to the preceding.

4.2. **Completed research.**

G-bundles. Here, many of the methods used in the compact and covering space cases require adjustment. To form a closed, G-invariant space analogous to L_N, we define, for $F \in L^2(M)$,

$$\langle F \rangle = \overline{\left\{ \sum_{k}^{\text{finite}} \alpha_k F(\cdot\, t_k) \mid \alpha_k \in \mathbb{C},\ t_k \in G \right\}}^{L^2(M)}.$$

First, it is not clear what the G-dimension of such a space should be. Second, $\langle \bar\partial \chi f \rangle$ need not be smooth, therefore taking an arbitrary $\phi \in \langle \bar\partial \chi f \rangle$ and solving $\Box u = \phi$ will not necessarily yield a smooth correction function u. Third, it could be that many members of $\langle \chi f \rangle$ are smooth in the boundary and we will be unable to argue as usual that the holomorphic function we construct is nonzero.

In [P2], the first problem is solved by a Paley-Wiener-type result on G. It turns out that if $F \in L^2(M)$ with compact support, then $\dim_G \langle F \rangle = \infty$. The second difficulty is circumvented by constructing smooth invariant subspaces of $\langle \bar\partial \chi f \rangle$ with arbitrarily large G-dimensioins. The third problem is deferred by making a technical definition. Roughly, the action of G on M is *amenable* if $R_\Delta \chi f \notin C^\infty(\bar M)$ for all nonzero $\Delta \in C^\infty(G)$.

In the case that the G-action on M is amenable, a weak version of the Levi problem is solved in [P2]. It is established that there are many square-integrable holomorphic functions on M and that each of these functions is nonsmooth in bM. In particular, these functions do not have holomorphic extensions beyond bM as Levi asked.

4.3. **Future directions.** First, a detailed study of amenability is in order. For example, take $G = \mathbb{R}^n$ and construct the tube $\mathcal{T}$ over $S^1 \times G$. It is clear that there is a holomorphic, free G-action on $\mathcal{T}$ and so $G \to \mathcal{T} \to \mathcal{B}$ satisfies all the assumptions of our theorems. Simple integral estimates in [P2] show that the action is amenable (without the S^1 factor, amenability fails). This should be generalized to the case in which G is an arbitrary unimodular Lie group and more examples should be sought *e.g.* the tubes over real G-bundles rather than just products $S^1 \times G$. An important goal along these lines is to apply our results to the gauged Stein G-complexifications of [HHK]. Furthermore, the idea first used in [P2] of applying the G-Fredholm property in tandem with the noncommutative Paley-Wiener property to measure spaces is in itself attractive and seems a fertile direction for future research.

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